Rational Exponents
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1.6

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Negative Exponents and the Quotient Rule

Negative Exponents

Suppose that a is a nonzero real number and n any integer.

$$a^{-n} = \frac{1}{a^{n}}, \qquad a^{n} = \frac{1}{a^{-n}}, \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

Negative Exponents and the Quotient Rule

Example 1 : Using the Definition of a negative exponent.

Evaluate each expression. In parts a,b,c,d and (e), write the expression without negative exponents. Assume all variables represent nonzero real numbers.

a) 4^{-2}

b) -4^{-2}

c) $\left(\frac{2}{5}\right)^{-3}$

d) $(xy)^{-3}$

e) xy^{-3}

Negative Exponents and the Quotient Rule

Quotient Rule

Suppose that m and n are integer and a is a nonzero real number.

$$\frac{a^m}{a^n} = a^{m-n}, \qquad \frac{a^m}{a^{-n}} = a^{m+n}$$

Negative Exponents and the Quotient Rule

Homework 1 : Using the Quotient Rule.

Simplify each expression. Assume all variables represent nonzero real numbers.

a) $\frac{12^5}{12^2}$

$b)\frac{a^5}{a^{-8}}$

$c)\ \frac{16m^{-9}}{12m^{11}}$

$d)\frac{25r^7z^5}{10r^9z}$

Negative Exponents and the Quotient Rule

Example 2 : Using the Rules for Exponents.

Simplify each expression. Write answers without negative exponents. Assume all variables represent nonzero real numbers.

a) $3x^{-2}(4^{-1}x^{-5})^{2}$

b) $\dfrac{12p^{3}q^{-1}}{8p^{-2}q}$

c) $\dfrac{(3x^{2})^{-1}(3x^{5})^{-2}}{(3^{-1}x^{-2})^{2}}$

Rational Exponents

Definitions and Rules for Exponents

Suppose that r and s represent rational numbers. The results here are valid for all positive numbers a and b.

Product rule 1 $a^r \cdot a^s = a^{r+s}$

Product rule 2 $(a^r)^s = a^{rs}$

Quotient rule $\frac{a^r}{a^s} = a^{r-s}$, $(ab)^r = a^r b^r$, $(\frac{a}{b})^r = \frac{a^r}{b^r}$

Negative exponent $a^{-r} = \frac{1}{a^r}$

Rational Exponents

The expression $a^{\frac{1}{n}}$

$a^{\frac{1}{n}}, n\ Even$

If n is an even positive integer, and if $a > 0$, then $a^{\frac{1}{n}}$ is the positive real number whose nth power is a, That is $(a^{\frac{1}{n}})^n = a$.

(In this case $a^{\frac{1}{n}}$ is the principle nth root of a)

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Rational Exponents

$a^{\frac{1}{n}}, n\ Odd$

If n is an odd positive integer, and a is any nonzero real number, then $a^{\frac{1}{n}}$ is the positive or negative real number whose nth power is a, That is $(a^{\frac{1}{n}})^n = a$. For all positive integers n, $0^{\frac{1}{n}} = 0$

Rational Exponents

Homework 2 : **Using the definition of $a^{\frac{1}{n}}$**

Evaluate each expression.

a) $36^{\frac{1}{2}}$

b) $-100^{\frac{1}{2}}$

c) $-(225)^{\frac{1}{2}}$

d) $625^{1/4}$

e) $(-1296)^{\frac{1}{4}}$

f) $-1296^{\frac{1}{4}}$

g) $(-27)^{\frac{1}{3}}$

h) $-32^{\frac{1}{5}}$

Rational Exponents

The expression $a^{\frac{m}{n}}$

Let m be any integer, n be any positive integer , and a be any real number for which $a^{\frac{1}{n}}$ is a real number.

$$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$$

Rational Exponents

Example 3 : **Using the definition of** $a^{\frac{m}{n}}$

Evaluate each expression.

$a)\ 125^{\frac{2}{3}}$

$b) 32^{\frac{7}{5}}$

$c)\ -81^{\frac{3}{2}}$

$d)\ (-27)^{2/3}$

$e) 16^{\frac{-3}{4}}$

$f) (-4)^{\frac{5}{2}}$

Homework 3 : Using the Rules for exponents

Simplify each expression. Assume all variables represent positive real numbers.

a) $\dfrac{27^{\frac{1}{3}} \cdot 27^{\frac{5}{3}}}{27^{3}}$

$b) 81^{\frac{5}{4}} \cdot 4^{-\frac{3}{2}}$

Rational Exponents

$c)\, 6y^{\frac{2}{3}} \cdot 2y^{\frac{1}{2}}$

$d)\, \left(\frac{3m^{5/6}}{y^{\frac{3}{4}}}\right)^{2} \left(\frac{8y^{3}}{m^{6}}\right)^{\frac{2}{3}}$

$e)\, m^{\frac{2}{3}}\left(m^{\frac{7}{3}} + 2m^{\frac{1}{3}}\right)$

Example 4 : Factoring Expressions with Negative or Rational Exponents

Factor out the least power of the variable or variable expression. Assume all variables represent positive real numbers.

a) $12x^{-2} - 8x^{-3}$

b) $4m^{\frac{1}{2}} + 3m^{\frac{3}{2}}$

c) $(y-2)^{\frac{-1}{3}} + (y-2)^{\frac{2}{3}}$

Complex Fractions Revisited

Negative exponents are sometimes used to write complex fractions.

Homework 4 : Simplifying a Fraction with Negative Exponents

Simplify $\frac{(x+y)^{-1}}{x^{-1}+y^{-1}}$. Write the results with only positive exponents.

Radical Notation

In this section we used rational exponents to express roots. An alternative notation for roots is radical notation.

Radical Notation for $a^{1/n}$:

Suppose that a is areal number, n is a positive integer, and $a^{1/n}$ is a real number.

$$a^{1/n} = \sqrt[n]{a}$$

Radical Notation for $a^{m/n}$:

Suppose that a is areal number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number.

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

Radical Notation

In the radical $\sqrt[n]{a}$, the symbol $\sqrt[n]{\quad}$ is a radical symbol

The number a is the radicand, and n is the index. We use the familiar notation $\sqrt{a}$ instead of $\sqrt[2]{a}$ for the square root.

For even of n(square roots, fourth roots, and so on),

when a is positive, there are two nth roots, one positive and one negative.

In such cases, the notation $\sqrt[n]{a}$ represents the positive root, the principal nth root.

We write the negative root as $-\sqrt[n]{a}$.

Example 5 : Evaluating Roots

Write each root using exponents and evaluate.

a) $\sqrt[4]{16}$ b) $-\sqrt[4]{16}$ c) $\sqrt[5]{-32}$ d) $\sqrt[3]{1000}$

e) $\sqrt[6]{\frac{64}{729}}$ f) $\sqrt[4]{-16}$

SOLUTION

a) $\sqrt[4]{16} = 16^{1/4} = 2$

b) $-\sqrt[4]{16} = -16^{1/4} = -2$

c) $\sqrt[5]{-32} = (-32)^{1/5} = -2$

d) $\sqrt[3]{1000} = (1000)^{\frac{1}{3}} = 10$

e) $\sqrt[6]{\frac{64}{729}} = (\frac{64}{729})^{\frac{1}{6}} = \frac{2}{3}$

f) $\sqrt[4]{-16} = (-16)^{1/4}$ *is not a real number.*

Homework 5 : Converting from Rational Exponents to Radicals

Write in radical form and simplify. Assume all variable expressions represent positive real numbers.

a) $8^{\frac{2}{3}}$ $\quad$ $b)(-32)^{\frac{4}{5}}$ $\quad$ $c)\ -16^{\frac{3}{4}}$ $\quad$ $d)\ x^{\frac{5}{6}}$

$e)\ 3x^{\frac{2}{3}}$ $\quad$ $f)2p^{\frac{1}{2}}$ $\quad$ $g)(3a+b)^{\frac{1}{4}}$

SOLUTION

a) $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4 \quad or \quad 8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$

$b)(-32)^{\frac{4}{5}} = \sqrt[5]{(-32)^4} = \sqrt[5]{1048576} = 16 \quad or \quad (-32)^{\frac{4}{5}} = (\sqrt[5]{-32})^4 = (-2)^4 = 16$

Radical Notation

SOLUTION

$c)\ -16^{\frac{3}{4}} = -\left(\sqrt[4]{16}\right)^3 = -2^3 = -8$

$d)\ x^{\frac{5}{6}} = \sqrt[6]{x^5}$

$e)\ 3x^{\frac{2}{3}} = 3\sqrt[3]{x^2}$

$f)\ 2p^{\frac{1}{2}} = 2\sqrt{p}$

$g)\ (3a+b)^{\frac{1}{4}} = \sqrt[4]{(3a+b)}$

Radical Notation

CAUTION

It is not possible to distribute exponents over a sum, so in Homework 5(g), $(3a+b)^{\frac{1}{4}} \neq (3a)^{\frac{1}{4}} + b^{\frac{1}{4}}$

$$\sqrt[n]{x^n + y^n} \neq x + y$$

Radical Notation

Example 6 : Converting from Radicals to Rational Exponents

Write in exponential form. Assume all variable expressions represent positive real numbers.

$a)\ \sqrt[4]{x^5}$ $\quad b)\sqrt{3y}$ $\quad c)\ 10(\sqrt[5]{z})^2$ $\quad d)\ 5\sqrt[3]{(2x^4)^7}$

$e)\ \sqrt{p^2+q}$

SOLUTION

$a)\ \sqrt[4]{x^5} = x^{\frac{5}{4}}$ $\quad b)\sqrt{3y} = (3y)^{\frac{1}{2}}$

$c)\ 10(\sqrt[5]{z})^2 = 10z^{\frac{2}{5}}$ $\quad d)\ 5\sqrt[3]{(2x^4)^7} = 5(2x^4)^{\frac{7}{3}} = 5\cdot 2^{\frac{7}{3}}\, x^{\frac{28}{3}}$

$e)\ \sqrt{p^2+q} = (p^2+q)^{\frac{1}{2}}$

Evaluating $\sqrt[n]{a^n}$

Evaluating $\sqrt[n]{a^n}$

suppose that a is a real number, If n is an even positive integer, then

$\sqrt[n]{a^n} = |a|$

Example: $\sqrt{(-9)^2} = |-9| = 9, \ \sqrt{13^2} = |13| = 13$

suppose that a is a real number, If n is an odd positive integer, then

$\sqrt[n]{a^n} = a$

Example: $\sqrt[5]{2^5} = 2, \qquad \sqrt[3]{(-8)^3} = -8$

Homework 6. Using Absolute Value to simplify Roots

Simplify each expression.

a) $\sqrt{p^4}$

b) $\sqrt[4]{p^4}$

c) $\sqrt{16m^8r^6}$

d) $\sqrt[6]{(-2)^6}$

e) $\sqrt[5]{m^5}$

f) $\sqrt{(2k+3)^2}$

g) $\sqrt{x^2-4x+4}$

Rules for Radicals

Suppose that a and b represent real numbers, and m and n represent positive integers for which the indicated roots are real numbers.

Rule

Product rule

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

Quotient rule

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}\ , (b \neq 0)$$

Power rule

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Example 7. Simplifying Radical Expressions

Simplify. Assume all variable expression represent positive real numbers.

a) $\sqrt{6} \cdot \sqrt{54}$

b) $\sqrt[3]{m} \cdot \sqrt[3]{m^2}$

c) $\sqrt{\frac{7}{64}}$

d) $\sqrt[4]{\frac{a}{b^4}}$

e) $\sqrt[7]{\sqrt[3]{2}}$

f) $\sqrt[4]{\sqrt{3}}$

Simplified Radicals

Simplified Radicals:

An expression with radicals is simplified when all of the following conditions are satisfied.

1. The radicand has no factor raised to a power greater than or equal to the index. الاسس داخل الجذر تكون اقل من الدليل
2. The radicand has no fractions لا توجد كسور داخل الجذر
3. No denominator contain a radical لا يوجد جذر في المقام
4. Exponents in the radicand and the index of the radical have greatest common factor 1 العامل المشترك بين الاس والدليل ١
5. All indicated operations have been performed (if possible)

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Homework 7.

Simplify each radical.

a) $\sqrt{175}$

$b) -3\sqrt[5]{32}$

$c)\ \sqrt[3]{81x^5y^7z^6}$

Operations on Radicals

Radicals with the same radicand and the same index, such as

$3\sqrt[4]{11pq}$ *and* $-7\sqrt[4]{11pq}$, **are like radicals,**

On the other hand, examples of unlike radicals are as follows:

$2\sqrt{5}$, *and* $2\sqrt{3}$ ***radicands are different***

$2\sqrt{3}$ *and* $2\sqrt[3]{3}$ ***indexes are different.***

We add or subtract like radicals by using distributed property. Only like radicals can be combined. Sometimes we need to simplify radicals before adding or subtracting.

Example 8. Adding and Subtracting Radical

Add or subtract, as indicated. Assume all variable expression represent positive real numbers.

a) $3\sqrt[4]{11pq}+\left(-7\sqrt[4]{11pq}\ \right)$

b) $\sqrt{98x^3y}+3x\sqrt{32xy}$

c) $\sqrt[3]{64m^4n^5}-\sqrt[3]{-27m^{10}n^{14}}$

Rules for Radicals

Homework 8. ***Simplifying Radicals***

Simplify each radicals. Assume all variables represent positive real numbers.

$a)\ \sqrt[6]{3^2}$

$b)\ \sqrt[6]{x^{12}y^3}$

$c)\ \sqrt[9]{\sqrt{6^3}}$

Rules for Radicals

Example 9. *Multiplying Radical Expressions*

Find each product

$a)\ (\sqrt{7}-\sqrt{10})(\sqrt{7}+\sqrt{10})$

$b)(\sqrt{2}+3)(\sqrt{8}-5)$

Rationalizing Denominators

The third condition for a simplified radical requires that no denominator contain a radical.

We achieve this by rationalizing the denominator- that is, multiplying by a form of 1.

Rationalizing Denominators

Homework 9: Rationalizing Denominators

Rationalize each denominator $a)\ \frac{4}{\sqrt{3}}$ $b)\ \sqrt[4]{\frac{3}{5}}$

Rationalizing Denominators

Example 10: Simplifying Radicals Expressions with Fractions Simplify each Expression. Assume all variables represent positive real numbers

$$a)\ \frac{\sqrt[4]{xy^3}}{\sqrt[4]{x^3y^2}}$$

$$b)\ \sqrt[3]{\frac{5}{x^6}} - \sqrt[3]{\frac{4}{x^9}}$$

Rationalizing a Binomial Denominators

Homework 10: Rationalizing a Binomial Denominator

Rationalize the denominator of $\frac{1}{1-\sqrt{2}}$